$$X_t = M_t - B_t = \begin{cases} \alpha - W_{t-s} & \tau > t \\ \tilde{M}_{t-\tau} \tilde{W}_{t-\tau} & \tau \le t \end{cases}$$

Y

Case 1 for simplicity :—

$$B_t = \begin{cases} B_s + W_{t-s} & \tau > t \\ \quad (\alpha + W_{t-s} < 0) & \\ \tilde{W}_{t-\tau} & \tau \le t \end{cases}$$

$$Y_t = \begin{cases} \alpha - W_{t-s} & t < \tau \\ |\tilde{W}_{t-\tau}| & t \geq \tau \end{cases}$$

24-09-2009

Recap: Levy's identity

B - std 1-dim B.M., $X = M - B$, $Y = |B|$

$\mathcal{F}_s = \sigma\{B_u | u \le s\}$

$W, \tilde{W}$ - iid 1-dim std BMs

Fix $s < t$, $\alpha \geq 0$, $\tau = \inf\{u | W_u = \alpha\} + s$

Then we showed that

$$X_t \big| \{\mathcal{F}_s, X_s = \alpha\} \overset{d}{=} (\alpha - W_{t-s}) 1_{\tau > t} + (\tilde{M}_{t-\tau} - \tilde{W}_{t-\tau}) 1_{\tau \le t}$$

$\overset{!!}{U}$

$$Y_t \big| \{\mathcal{F}_s, Y_s = \alpha\} \overset{d}{=} (\alpha - W_{t-s}) 1_{\tau > t} + |\tilde{W}_{t-\tau}| 1_{\tau \le t}$$

$\overset{!!}{V}$

Condition on τ; $\tilde{W}|_\tau$ is still a B.M.

Then if $\tau \geq t$, $U = (\alpha - \tilde{W}_{t-s}) = V$

if $\tau < t$, $U = (\tilde{M} - \tilde{W})_{t-\tau}$, $V = |\tilde{W}|_{t-\tau}$ } have same distributⁿ for fixed $t-\tau$ (by exer.)

i.e $U|_{\tau=u} \stackrel{d}{=} V|_{\tau=u} \quad \forall u$

$\Rightarrow \quad U \stackrel{d}{=} V$

Thus, $X_t|\{\mathcal{F}_s, X_s = \alpha\} \stackrel{d}{=} Y_t|\{\mathcal{F}_s, Y_s = \alpha\}$

and this distribution depends on α but not on the conditioned value of $\{X_u, u<s\}$ or $\{Y_u, u<s\}$

|| Note that if μ_y is free of z, then
$X|(Y=y_1, Z=z_0) \sim \mu_y \Rightarrow X|_{Y=y} \sim \mu_y$ ||

Hence $X_t|\{X_u, u \leq s, X_s = \alpha\} \stackrel{d}{=} Y_t\{Y_u, u \leq s, Y_s = \alpha\}$

& distribution depends only on α.

In particular, if $t_1 < \dots < t_n$ then

$$X_{t_n}|(X_{t_1}=\alpha_1 \dots X_{t_{n-1}} = \alpha_{n-1}) \stackrel{d}{=} Y_{t_n}|(Y_{t_1}=\alpha_1, \dots Y_{t_{n-1}} = \alpha_{n-1})$$

inductively
$\Rightarrow \quad (X_{t_1}, \dots X_{t_n}) \stackrel{d}{=} (Y_{t_1}, \dots, Y_{t_n})$

Since X, Y are continuous, $\boxed{X \stackrel{d}{=} Y}$

Remarks:

1) The conditional law $X_t \mid \{X_u, u \le s\}$ depends on X_s but not on X_u, $u < s$ (This is called Markov property)

Further, this distribution depends on $t-s$ and not on s, t, separately. This is called time homogeneity. (Check this!)

This distribution, denote it $\mu_{X_s}^{t-s}$ is called the transition measure of the Markov process X.

2) In addition to $M-B \overset{d}{=} |W|$, Levy showed that M is a function of $M-B$. In fact, that $(M_u)_{u \le t}$ is a function of $(M_u - B_u)_{u \le t}$

→ Write $M = \phi(M-B)$

where $\phi : \underset{\text{non negative}}{C[0,\infty)} \longrightarrow \underset{\text{non decreasing}}{C[0,\infty)}$

Levy's identity: $M - B \overset{d}{=} |W|$

$$\Rightarrow (M-B, \phi(M-B)) \overset{d}{=} (|W|, \phi(W))$$

$$\Rightarrow (M-B, M) \overset{d}{=} (|W|, l_0^W)$$

where $l_0^W := \phi(|W|)$.

This means for W - std 1- dim BM.

We have a process l_0^W with the properties

a) l_0^W is adapted to $\mathcal{F}_\cdot$ = Filtration generated by W

(b) l_0^W is non decreasing, cts a.s.

(c) If $W_u \neq 0$, $\forall u \in [s,t)$ then $l_0^W(s) = l_0^W(t)$.

l_0^W is called the local time of W at level 0.
(measures time spent at 0.)

Sec 17: Stochastic integration of L^2 function.

Suppose $g \in C^1[0,1]$. Then for any $f \in C$,

$$\int_0^1 f\,dg := \int_0^1 f(t) g'(t)\,dt.$$

Recall: If $f, g \in C^1$

$$\int_0^1 f\,dg = f(1)g(1) - f(0)g(0) - \int_0^1 g\,df.$$

If $g \in C^0$ but $g \notin C^1$, then R.H.S makes sense for $f \in C^1$

Thus we define

$$\int_0^1 f\,dg := fg\Big|_0^1 - \int_0^1 g\,df \quad \text{for } f \in C^1$$

Objective of this section is to integrate fns against B.M.

1) $(B_t)_{0 \le t \le 1}$ std 1-dim B.M.

Define $\int_0^1 f(t)\, dB(t) = f(1)B(1) - f(0)B(0) - \int_0^1 B(t) f'(t)\, dt$, for $f \in C^1$

Basic requirements: At least we would like

a) $\int_0^1 (\alpha f + \beta g)\, dB(t)$

$= \alpha \int_0^1 f\, dB + \beta \int_0^1 g\, dB.$

(b) $\int_0^1 1_{[0,t)}(s)\, dB(s) = B(t) - B(0).$

Clearly it can be proved easily.

(2) Preliminaries: $l^2 = \{(x_1, x_2 \cdots) / x_i \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 < \infty\}$ is a Hilbert space.

with $\langle \underline{x}\ \underline{y} \rangle = \sum_{i=1}^{\infty} x_i y_i$.

Let $a_1, a_2 \cdots$ be iid $N(0,1)$ on some $(\Omega, \mathcal{F}, P)$

1) $\underline{a} = (a_1, a_2 \cdots) \notin l^2$ (a.s.)

(clearly $a_n \not\to 0$, $n \to \infty$ $\because$ a_is are iid r.v.).

2) Fix $\underline{X} \in l^2$. Then

$$\sum_{i=1}^{\infty} a_i x_i \text{ converges a.s.}$$

($\because$ Note that $E(a_i x_i) = 0$

$$E[(a_i x_i)^2] = x_i^2$$

$$\Rightarrow \sum E[(a_i x_i)^2] < \infty.)$$

Q: If $(b_1, \cdots)$, $b_i \in \mathbb{R}$, does $\sum b_i x_i$ converges $\forall\ \underline{x} \in l^2$?

Let $L(\underline{x}) = \sum b_i x_i$ If $L(\underline{x})$ cgs $\forall\ \underline{x} \in l^2$

then $L_n(\underline{x}) = \sum_{i=1}^{n} x_i b_i$

clearly $\{L_n(\underline{x})\}$ is a bdd sequence for each $\underline{x}$

$\therefore$ by uniform bddness principle

$$|L(\underline{x})| \leq M \|x\|$$

$\therefore$ by R.R.T. i.e if $L : l^2 \to R$ is cts bdd linear ~~fnal~~ functional then $\exists\ \underline{Y} \in l^2$ st

$$L(\underline{x}) = \langle \underline{x}, \underline{Y} \rangle.$$

$\therefore$ we have

(3) For a.e. ω, $\exists$ $\underline{x} \in \ell^2$ (depending on ω) such that $\sum_{i=1}^{\infty} a_i(\omega) x_i$ does not converge

($\because$ if $\sum a_i(\omega) x_i$ cgs $\forall$ $\underline{x} \in \ell^2$ then $(a_1(\omega), \dots) \in \ell^2$.)

2.) Now consider $L^2[0,1]$. Fix an O.N.B. say Haar basis $\{h_{n,k}\}_{1 \le k \le 2^n,\ n \ge 1}$

Then if $f \in L^2[0,1]$, we can write

$$f = \sum_{n,k} \hat{f}_{n,k} h_{n,k} \quad \text{where } \hat{f}_{n,k} = \langle f, h_{n,k} \rangle$$

$$= \int_0^1 f h_{n,k}$$

$$\hat{f} = (\hat{f}_{1,1}, \hat{f}_{2,1}, \dots) \in \ell^2[0,1]$$

Then $f \to \hat{f}$ is an isomorphism from $L^2[0,1]$ onto ℓ^2.

That is $\langle f, g \rangle_{L^2[0,1]} = \sum \hat{f}_{n,k} \hat{g}_{n,k} = \langle \hat{f}, \hat{g} \rangle_{\ell^2}$

Now recall that

$$B(t) = \sum_{n,k} a_{n,k} \int_0^t h_{n,k} \quad [a_{n,k} \text{ iid } N(0,1)]$$

So formally $B'(t) = \sum a_{n,k} h_{n,k}$

Thus we define

$$\int_0^1 f\,dB := \sum_{n,k} a_{n,k}\,\hat{f}_{n,k} \longrightarrow \text{cgs a.s. (for fixed } f \in L^2[0,1])$$

29-9-2009

RECAP

2) $L^2[0,1]$ $\{h_{n,k} \mid n \geq 1, k \leq 2^n\} \rightarrow$ Haar O.N.B.

$f \in L^2[0,1]$ can be written

$$f = \sum_{n,k} \hat{f}_{n,k}\, h_{n,k} \qquad \hat{f}_{n,k} = \langle f, h_{n,k} \rangle_{L^2[0,1]}$$

If $\hat{f} = (\hat{f}_{11}, \hat{f}_{21}, \hat{f}_{22} \cdots)$ then $f \rightarrow \hat{f}$ is an isomorphism between $L^2[0,1]$ & l^2

i.e $\langle f, g \rangle_{L^2[0,1]} = \langle \hat{f}, \hat{g} \rangle_{l^2}$.

Now recall B.M in 1-dim run for time 1 is

is $B(t) = \sum_{n,k} a_{n,k} \int_0^t h_{n,k}$ and hence formally

$$\text{"}\frac{dB(t)}{dt} = \sum_{n,k} a_{n,k}\, h_{n,k}\text{"}.$$

For fixed $f \in L^2[0,1]$, $\hat{f} \in l^2$ and hence

$\sum_{n,k} \hat{f}_{n,k}\, a_{n,k}$ converges and hence this can be regarded as "$\langle f, B' \rangle_{L^2[0,1]}$"

Define. $$\int_0^1 f(t)\,dB := \sum_{n,k} a_{n,k}\,\hat{f}_{n,k}$$

Note:—

1) For fixed f, $\int_0^t f(t)\,dB(t)$ exists a.s. (in ω)

2) For a.e. (ω) then $\exists\ f \in L^2[0,1]$ s.t
$\sum \hat{f}_{a_{n,k}} a_{n,k}$ does not converge.

3) For fixed $f_1, f_2, \cdots f_{k_1} \in L^2[0,1]$

$$\left(\int_0^1 f_1\,dB, \cdots\cdots, \int_0^1 f_{k_1}\,dB\right) \sim N_{k_1} \sim (0, \Sigma)$$

$$\Sigma_{ij} = \text{cov}\left(\int_0^1 f_i\,dB, \int_0^1 f_j\,dB\right)$$
$$= \langle f_i, f_j \rangle_{L^2[0,1]}$$

4) Exercise. If $f = 1_{[0,t]}$, $t \leq 1$

Then show that $\int_0^1 f\,dB = B(t) - B(0)$

Remark:— Let H be any separable H.S.
Let $\varphi_1, \varphi_2 \cdots$ be any O.N.B. of H Then
any $f \in H$ can be written as $f = \sum_n \hat{f}_n \varphi_n$
$\hat{f}_n = \langle f, \varphi_n \rangle$..

→ If $\hat{f} = (\hat{f}_1, \hat{f}_2, \cdots)$ then $f \to \hat{f}$ is an isomorphism
from H onto ℓ^2

→ In particular, $\langle f, g \rangle_H = \langle \hat{f}, \hat{g} \rangle_{\ell^2}$

|| Note in an infinite dimensional, one cannot define, translational invariant measure. ||

Now let $a_1, a_2 \cdots$ be iid $N(0,1)$ on some $(\Omega, \mathcal{F}, P)$

Let "$G = \sum a_n \varphi_n$" (this series does not converge unless $\dim(H) < \infty$.)

$\mathbb{R}^d$, $\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$, $a_i \overset{iid}{\sim} N(0,1)$

For $X \in \mathbb{R}^d$, $\langle \underline{a}, \underline{x} \rangle \sim N(0, \|x\|^2)$

If $x_1, \cdots x_k \in \mathbb{R}^d$ $(\langle a, x_1\rangle, \cdots \langle a, x_k\rangle)$ is jointly Gaussian mean 0, variance$((\langle x_i, x_j\rangle))_{i,j \le k}$.

However for any $f \in \mathcal{H}$, define the r.v. $G_f =$ "$\langle G, f\rangle$" $= \sum a_n \hat{f}_n$ Then for any $f_1, \cdots f_k \in H$ $G_{f_1} \cdots G_{f_k}$ are jointly gaussian with zero mean and covariance $((\langle f_i, f_j\rangle))_{i,j \le k}$.

In other words the map
$$\mathcal{H} \longrightarrow L^2(\Omega, \mathcal{F}, P)$$
$$f \longrightarrow G_f.$$

This is an isomorphism (Linear, $\langle f, g\rangle_{\mathcal{H}} = \mathrm{cov}(G_f, G_g)$) into $L^2[\Omega, \mathcal{F}, P]$.

3.) For a step function

$$f = \sum_{k=1}^{m} c_k \, 1_{[t_{k-1}, t_k)}$$

$$0 = t_0 < t_1 < \cdots < t_m \leq 1$$

$c_k \in \mathbb{R}$.

define $I_f = \int_0^1 f \, dB = \sum_{k=1}^{m} c_k \left[B(t_k) - B(t_{k-1}) \right]$

<u>Observation</u> If f, g are step functions then I_f, I_g are jointly gaussian.

$E[I_f] = 0 = E[I_g]$ and

$cov(I_f, I_g) = \langle f g \rangle_{L^2[0,1]}$

Thus I: Step functions $\subseteq L^2[0,1]$

$\longrightarrow L^2[\Omega, \mathcal{F}, P)$ is an isometry.

(Prob. sp. where B is defined)

Step functions dense in $L^2[0,1]$, hence I extends

(not onto) into $L^2(\Omega, \mathcal{F}, P)$ i.e if $f \in L^2[0,1]$, find $f_n \in L^2[0,1]$

Step fns $f_n \xrightarrow{L^2[0,1]} f$, the $I_f = L^2$ - lim of I_{f_n}

Exercise* check that definitions (1), (2) (3) all agree

((1) is defined only for C^1 fns, (2), (3) are for L^2 fns.)

Sec 18: Fractional dimensions.

Minkowski dimension:– Let (E, d) be a bounded metric space (i.e $\exists\ M > 0$ s.t $d(x,y) \leq M$, $\forall x, y \in E$.)

For $\varepsilon > 0$, let N_ϵ = minimal k s.t $\exists$ $A_1, A_2 \cdots A_k \subseteq E$ s.t $E \subseteq \bigcup_{j=1}^{k} A_j$ and $\mathrm{dia}(A_j) \leq \varepsilon$ $\forall j = 1, 2, \dots k$.

$$(\mathrm{dia}(A) = \sup_{x,y \in A} (d(x,y)))$$

Then define $\underbrace{\dim_M (E)}_{(\text{Minkowski dimension})} = \lim\limits_{\varepsilon \downarrow 0} \dfrac{\log N_\epsilon}{\log (1/\varepsilon)}$ if limit exists.

In general, we define, $\overline{\dim}_M (E) = \limsup\limits_{\epsilon \downarrow 0} \dfrac{\log N_\epsilon}{\log(1/\varepsilon)}$

$$\underline{\dim}_M (E) = \liminf_{\varepsilon \downarrow 0} \frac{\log N_\epsilon}{\log (1/\varepsilon)}$$

Examples:

1) $E = [0,1]^{d}$, d = usual metric from $\mathbb{R}^m$

Fix $\epsilon > 0$. Then let $A_{(k_1, k_2, \cdots k_m)}$

$$= \left[\frac{k_1 - 1}{\sqrt{d}}\epsilon, \frac{k_1}{\sqrt{d}}\epsilon\right] \times \cdots \left[\frac{k_d - 1}{\sqrt{d}}\epsilon, \frac{k_d}{\sqrt{d}}\epsilon\right]$$

Then $\mathrm{dia}(A_{k_1, \dots k_d}) = \epsilon$

and $\bigcup\limits_{\substack{(k_1, \cdots k_m) \\ 1 \leq k_i \leq \sqrt{m}/\varepsilon}} A_{(k_1, \cdots k_d)} \supseteq E$

Hence
$$N_\epsilon \le \left(\frac{\sqrt{d}}{\epsilon}\right)^d$$

$\therefore$
$$\frac{\log N_\epsilon}{\log(1/\epsilon)} = \frac{d\log\sqrt{d} + d\log(1/\epsilon)}{\log(1/\epsilon)}$$

$$\longrightarrow d \quad \text{as}$$

$$\Rightarrow \dim_M([0,1]^d) \le d$$

Consider the points $(2k_1\epsilon, 2k_2\epsilon, \cdots, 2k_d\epsilon)$,

$$0 \le k_j \le \frac{1}{2\epsilon}$$

Then distance between any two distinct ones among these is $> \epsilon$

$\Rightarrow$ $N_\epsilon \ge$ # of these points

$$= \left(\frac{1}{2\epsilon}\right)^d$$

$$\frac{\log N_\epsilon}{\log(1/\epsilon)} \ge \frac{d\log(1/2) + d\log(1/\epsilon)}{\log(1/\epsilon)}$$

$$\longrightarrow d$$

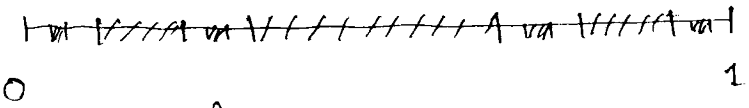
$$\Rightarrow \dim_M([0,1]^d) \ge d$$

Thus $\overline{\dim}_M = \underline{\dim}_M = \dim_M = \cancel{m}\ d$

01 - 10 - 2009

~~Main~~

Ex. 2 Let E be $1/3$ cantor set.



. . . and so-on

$$K_1 = [0, 1/3] \cup [2/3, 1]$$
$$\cup|$$
$$K_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$
$$\cup|$$
$$\vdots$$
$$K_n$$
$$\vdots$$

Then $E = \bigcap_n K_n$

Fix $\varepsilon > 0$, Let $3^{-n} \le \varepsilon < 3^{-n+1}$.

Then the 2^n intervals of length 3^{-n} that make up K_n form an ε-cover for E

Hence $N_\varepsilon \le 2^n$

$$\overline{\dim}_M(E) \le \lim_{n\to\infty} \frac{n \log 2}{(n-1)\log 3} = \frac{\log 2}{\log 3}$$

Lower bound :— Note that each of the 2^n intervals of K_n intersect E non trivially.

Pick $x_1, x_2 \cdots x_{2^{n-1}} \in E$.

One x_i in each of the 2^{n-1} intervals of K_{n-1}

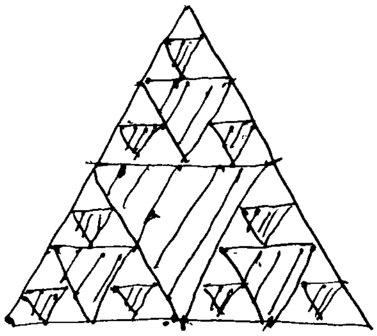
If $i \neq j$

$$|x_i - x_j| \geq 3^{-n+1} > \varepsilon$$

Hence $N_\varepsilon \geq 2^{n-1}$

$$\Rightarrow \underline{\dim}_M(E) \geq \lim_{n\to\infty} \frac{(n-1)\log 2}{\log 3} \to \frac{\log 2}{\log 3}$$

Ex* Find $\dim_M$ of the Sierpinski gasket



Eg 3 :— $E = \{0, 1, \frac{1}{2}, \frac{1}{3} \cdots\}$

Fix $\varepsilon > 0$.

If $\frac{1}{k(k+1)} < \varepsilon$ then, $1, \frac{1}{2}, \cdots \frac{1}{k}$

are at a distance $> \varepsilon$ from each other

Hence $N_\varepsilon \geq \frac{1}{\sqrt{\varepsilon}}$ (about)

$$\Rightarrow \dim_M(E) \geq \frac{1}{2}$$

Ex: $\left\{ \frac{1}{\log n} \right\}$

Hausdorff dimension:— (E, d) : bd metric space.

Fix $\alpha \geq 0$. For any $B \subseteq E$, define

$$H_\alpha(B) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_i |A_i|^\alpha \;\middle|\; \text{where } \{A_i\} \text{ is a } \delta\text{-cover for } B \right\}$$

$\Downarrow$

$(|A_i| = \text{diam } A_i \leq \delta \;\; \forall i)$

This is an increasing limit

Remark:— H_α is actually an outer measure, i.e

1) defined $\forall$ subsets $B \subseteq E$

2) $H_\alpha(\phi) = 0$

3) $B_1 \subseteq B_2 \Rightarrow H_\alpha(B_1) \leq H_\alpha(B_2)$

4) $H_\alpha(\bigcup_i B_i) \leq \sum_i H_\alpha(B_i)$

countable subadditivity.

|| Caratheodary condition: A is measurable if $H_\alpha(A \cap B) + H_\alpha(A^c \cap B) = H_\alpha(B) \;\; \forall B$ ||

5) H_α restricted to (---) becomes measure.

Observation: Consider $H_\alpha(E)$

Let $\alpha < \beta$

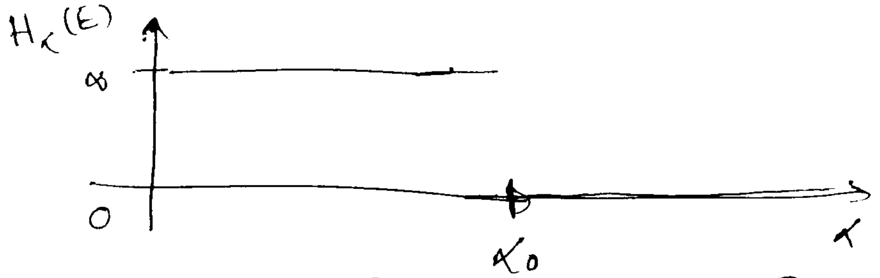
Suppose $\{A_i\}$ is a δ-cover of E

$$\sum_i |A_i|^\beta \le \delta^{\beta-\alpha} \sum_i |A_i|^\alpha$$

$$\inf_{\delta \text{ covers}} \sum |A_i|^\beta \le \delta^{\beta-\alpha} \inf_{\delta \text{ covers}} \sum |A_i|^\alpha$$

Let $\delta \to 0$. If $H_\alpha(E) < \infty$ then

$$H_\beta(E) = 0$$



Let $\alpha_0 = \sup\{\alpha \mid H_\alpha(E) = \infty\}$

$$H_\alpha(E) = \begin{cases} \infty & \text{if } \alpha < \alpha_0 \\ 0 & \text{if } \alpha > \alpha_0 \end{cases}$$

($H_{\alpha_0}(E)$ could be 0 or ∞ or in between)

Defn:— Hausdorff dimension

$$\dim_H(E) = \sup\{\alpha \mid H_\alpha(E) = \infty\}$$
$$= \inf\{\alpha \mid H_\alpha(E) = 0\}$$

In this case also the Hausdorff ~~[illegible]~~ dimension doesn't change for equivalent matrix.

Eg: 1 $[0,1]^d$. We know $H_d = C_d \cdot$ Lebesgue measure

$\longrightarrow$ non trivial measure.

In particular $0 < H_d([0,1]^d) < \infty$

$\Rightarrow \dim_H([0,1]^d) = d$

Eg: 2 $E = \frac{1}{3}$ Cantor set. Fix $\delta = 3^{-n}$ Let $\alpha \geq 0$.

We want to find $\inf_{\substack{\delta\text{-covers} \\ \text{of } E}} \sum_i |A_i|^\alpha$

An upper bound can be got by taking a particular δ-cover $\rightarrow \left\{ \begin{array}{l} 2^n \text{ intervals of length } 3^{-n} \\ \text{making up } K_n \end{array} \right.$

For this cover $\sum_i |A_i|^\alpha = 2^n 3^{-n\alpha}$

Hence $H_\alpha(E) = \lim_{\delta \downarrow 0} \inf \{ \quad \}$

$$\leq \lim_{n\to\infty} 2^n 3^{-n\alpha}$$

$$= \begin{cases} 0 & \text{if } 3^\alpha > 2 \text{ i.e. if } \alpha > \frac{\log 2}{\log 3} \end{cases}$$

$$\Rightarrow \dim_H(E) \leq \frac{\log 2}{\log 3}$$

Note That

$$E = \left\{ \sum_{k=1}^{\infty} \frac{x_k}{3^k} \,\middle|\, x_k = 0/2 \right\}$$

Remarks:— The idea in Eg 2 shows in general that $\dim_H(E) \leq \underline{\dim}_M(E)$

Reason: Fix α. Let $\delta > 0$. Then take the δ cover with minimal number of sets. So N_δ of them

Hence $\inf_{\delta\text{-covers}} \sum_i |A_i|^\alpha \leq \delta^\alpha N_\delta$

If $\alpha > \underline{\dim}_M(E)$ then $\exists\, \delta_j \to 0$ s.t $\delta_j^\alpha N_{\delta_j} \to 0$

$\Rightarrow H_\alpha(E) = 0$.

Eg 3 $E = \{0, 1, \frac{1}{2}, \cdots\}$

Fix $\delta > 0$ Let $A_0 = [0, \delta]$.

cover $1, \frac{1}{2} \cdots \frac{1}{[1/\delta]}$ each with an intervals of diameter 0.

$$\inf_{\delta \text{ covers}} \sum_i |A_i|^\alpha \leq \delta^\alpha \to 0 \text{ as } \delta \to 0$$

$\forall\, \alpha > 0$

$\Rightarrow H_\alpha(E) = 0 \quad \forall\, \alpha > 0$

$\Rightarrow \dim_H(E) = 0$.

Ex Remark: Countable stability of $\dim_H$

If $E = \bigcup_j E_j$ ← a countable.

then $\dim_H(E) = \sup_j \dim_H(E_j)$

Energy and Capacity :—

Consider $B_\alpha(0,1) = \{x \in \mathbb{R} / \|x\| \le 1\}$

$$\int_{-1}^{1} \frac{dx}{|x|^\alpha} \begin{cases} < \infty & \text{if } \alpha < 1 \\ = \infty & \text{if } \alpha \ge 1 \end{cases}$$

$$\int_{B_d(0,1)} \frac{dx}{\|x\|^\alpha} < \infty \quad \text{iff} \quad \alpha < d$$

In this way, the integral over unit ball $\frac{1}{\|x\|^\alpha}$ is finite for $\alpha < d$

$\Rightarrow$ d-dimension.

(E, d) bdd m.s. For a Borel p.m. μ on E, & $\alpha \ge 0$ define the α-energy of μ to be

1) $$\mathcal{E}_\alpha(\mu) = \int_E \int_E \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}.$$

2) Define $$\mathrm{Cap}_\alpha(E) = \frac{1}{\inf\limits_{\substack{\mu \text{ p.m.} \\ \text{on } E}} \mathcal{E}_\alpha(\mu)}$$

$\swarrow$

α - capacity of E

Examples: 6-10-2009

1) $E = [0,1]$ μ = uniform $\mathcal{E}_\alpha(\mu) = \int_0^1\int_0^1 \frac{dx\,dy}{|x-y|^\alpha}$

Fix x, $\int_0^1 \frac{dy}{|x-y|^\alpha}$

$$= \int_0^x \frac{dy}{(x-y)^\alpha} + \int_x^1 \frac{dy}{(y-x)^\alpha}$$

$$= \begin{cases} \infty & \text{if } \alpha \geq 1 \\ -\left.\frac{(x-y)^{-\alpha+1}}{-\alpha+1}\right|_0^x + \left.\frac{(y-x)}{-\alpha+1}\right|_x^1 & \text{if } \alpha < 1 \\ = \frac{x^{-\alpha+1}}{1-\alpha} + \frac{(1-x)^{-\alpha+1}}{(1-\alpha)} & \end{cases}$$

$$= \begin{cases} \infty & \alpha \geq 1 \\ \int_0^1 \frac{x^{-\alpha+1} + (1-x)^{-\alpha+1}}{(1-\alpha)}\,dx = \frac{2}{(1-\alpha)(2-\alpha)} & \alpha < 1 \end{cases}$$

As a consequence, $\mathrm{cap}_\alpha [0,1] > 0$ if $\alpha < 1$.

For $\alpha \geq 1$, it is true that $\mathrm{cap}_\alpha([0,1]) = 0$

To show that we must show that

$\mathcal{E}_\alpha(\mu) = \infty$ $\forall$ p.m. μ on $[0,1]$.

Suppose μ is absolutely cts say $d\mu(x) = f(x)dx$

Then $\mathcal{E}_\alpha(\mu) = \int_0^1\int_0^1 \frac{f(x)f(y)}{|x-y|^\alpha}\,dy\,dx.$

Fix x such that $f(x) > 0$. (Assume f is cts)

Consider

$$\int_0^1 \frac{f(y)\,dy}{|x-y|^\alpha} = \begin{cases} \infty & \text{if } \alpha \geq 1 \\ < \infty & \text{if } \alpha > 1 \end{cases}$$

Hence, $\mathcal{E}_\alpha(\mu) = \infty$ if $\alpha \geq 1$.

Ex*:- Show that for any probability measure μ on $[0,1]$, $\mathcal{E}_\alpha(\mu) = \infty$ if $\alpha \geq 1$

(Hint:- Consider $g \in L^1[0,1]$, then $\dfrac{\int_{x-\delta}^{x+\delta} g(y)\,dy}{2\delta}$ exists as $\delta \downarrow 0$ and is equal to $g(x)$ a.e.)

The example motivates

Defn:- Let $\dim_{\mathcal{E}}(E) = \sup\{\alpha \mid \mathrm{Cap}_\alpha(E) > 0\}$
$= \inf\{\alpha \mid \mathrm{Cap}_\alpha(E) = 0\}$

("energy dimension of E").

Defn makes sense because: Fix μ a p.m & consider $\mathcal{E}_\alpha(\mu) = \iint_{E\,E} \frac{d\mu(x)\,d\mu(y)}{d(x,y)^\alpha}$ as a fn of α.

$$\mathcal{E}_\alpha(\mu) = \underbrace{\iint_{d(x,y)\le 1} \frac{d\mu(x)\,d\mu(y)}{d(x,y)^\alpha}}_{\text{increasing in } \alpha.} + \underbrace{\iint_{d(x,y)>1} \frac{d\mu(x)\,d\mu(y)}{d(x,y)^\alpha}}_{\text{bounded by } 1}$$

Hence, $\mathcal{E}_\alpha(\mu) = \infty \implies \mathcal{E}_\beta(\mu) = \alpha \quad \forall \beta > \alpha.$

$\therefore$ if $\inf_\mu \mathcal{E}_\alpha(\mu) = \infty$ then $\inf_\mu \mathcal{E}_\beta(\mu) = \alpha \quad \forall \beta>\alpha$

Hence $\mathrm{cap}_\alpha(E) = 0$ then $\mathrm{cap}_\beta(E) = 0 \quad \forall \beta > \alpha$

Thus equality in the defn makes sense.

This is equal to Hausdorff dimension under fairly general conditions. But we will only prove that this provides an ~~upper~~ lower bound for Hausdorff dimension.

Lemma:— (E,d) : bdd m.s. Then

$$\dim_{\mathcal{E}}(E) \le \dim_H(E) \le \underline{\dim}_M(E)$$

Proof:— II^{nd} inequality was proved earlier.

$\underline{\mathrm{I}^{st}\text{ ineq}}$: — Fix $\alpha < \dim_{\mathcal{E}}(E)$

$\implies \mathrm{cap}_\alpha(E) > 0 \implies \exists\, \mu$ s.t $\mathcal{E}_\alpha(\mu) < \infty$

$_{\alpha}(E) > 0 \Rightarrow \dim_H(E) \geqslant \alpha.$

consider $H_\alpha(E) = \lim\limits_{\delta \downarrow 0} \inf\limits_{\delta \text{ covers}} \underline{\sum_j |A_j|^\alpha}$ $(+)$

Fix $\delta > 0$. Let $\{A_j\}$ be any δ-cover of E.

We know

$$\mathcal{E}_\alpha(\mu) < \infty$$

But $\infty > \mathcal{E}_\alpha(\mu) = \displaystyle\iint_{E\ E} \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}$

$$= \iint_{\bigcup_j A_j \ \bigcup_i A_i} \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}$$

$$= \sum_{i,j} \int_{A_j} \int_{A_i} \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}$$

$$\geqslant \sum_i \int_{A_i} \int_{A_i} \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}$$

$$\geq \sum_i \frac{\mu(A_i)^2}{|A_i|^\alpha}$$

Now consider $\sum_j |A_j|^\alpha$

$$\left(\sum_j |A_j|^\alpha\right)\left(\sum_j \frac{\mu(A_j)^2}{|A_j|^\alpha}\right)$$

$$\geq \sum_j \mu(A_j) \quad \text{(Cauchy Schwartz ineq)}$$

$$\geq 1 \quad (\because \cup A_j = E)$$

Therefore

$$\sum_j |A_j|^\alpha \geq \frac{1}{\sum_j \frac{\mu(A_j)^2}{|A_j|^\alpha}} \geq \frac{1}{\mathcal{E}_\alpha(\mu)}$$

$$\Rightarrow H_\alpha(E) = \lim_{\delta \downarrow 0} \inf_{\substack{\delta \\ \text{covers}}} \sum_j |A_j|^\alpha \geq \frac{1}{\mathcal{E}_\alpha(\mu)} > 0$$

$$\Rightarrow \dim_H(E) \geq \alpha.$$

This is true $\forall \alpha < \dim_\mathcal{E}(E)$

$$\Rightarrow \dim_H(E) \geq \dim_\mathcal{E}(E)$$

...ow there are some gaps.

Gap: Let $\{A_j\}$ be any δ-cover of E

Then $\{\bar{A}_j\}$ is also a δ cover of E (∵ $dia(\bar{A}_j) = dia(A_j)$)

& ~~hence~~ ∴ closed & hence Borel sets.

& $\sum_j |A_j|^\alpha = \sum |\bar{A}_j|^\alpha$

Next let $B_1 = \bar{A}_1$, $B_2 = \bar{A}_2 / \bar{A}_1$

... $B_K = \bar{A}_K / (\bar{A}_1 \cup \cdots \bar{A}_{K-1})$

Then $\{B_K\}$ is a δ cover of E & each B_K is a Borel set

$\sum |B_K|^\alpha \le \sum_j |A_j|^\alpha$

Thus from (⁎)

$$H_\alpha(E) = \lim_{\delta \downarrow 0} \inf_{\delta \text{ covers}} \sum_j |A_j|^\alpha$$

$$= \lim_{\delta \downarrow 0} \inf \left\{ \sum_j |A_j|^\alpha \,\middle|\, \begin{array}{l}\{A_j\} - \text{a } \delta \text{ cover, each } A_j \\ \text{is Borel, } A_i \cap A_j = \phi \text{ if } i \neq j\end{array} \right\}.$$

Example. $E = \frac{1}{3}$ Cantor set.

U.B

We showed that $\dim_M(E) = \frac{\log 2}{\log 3}$

$\Rightarrow \dim_H(E) \le \frac{\log 2}{\log 3}$

L.B Let $x_1, x_2 \dots$ be iid

$P\{x_1 = 0\} = \frac{1}{2} = P\{x_1 = 2\}$

Set $X = \sum_{k=1}^{\infty} \frac{x_k}{3^k}$

X takes values in E. If μ = distn of X

then μ is supported on E

Consider $\mathcal{E}_\alpha(\mu) = \iint_{E\,E} \frac{d\mu(x)\, d\mu(y)}{d(x,y)^\alpha}$

$= E\left[|X-Y|^{-\alpha}\right]$

where $X, Y \sim$ iid μ.

Let $L = \min\{k \mid x_k \ne y_k\}$

where $X = \frac{x_1}{3} + \frac{x_2}{3^2} + \dots$

$Y = \frac{y_1}{3} + \frac{y_2}{3^2} + \dots$

$$\therefore \; |X-Y| \le \frac{2}{3^L} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{3^{L-1}}$$

Similarly $\quad 3^{-L} \le |X-Y|$

$$\Rightarrow \quad E\left[3^{(L-1)\alpha}\right] \le E\left[|X-Y|^{-\alpha}\right] \le E\left[3^{\alpha L}\right]$$

$$\| $$

$$3^{-\alpha} E\left[3^{\alpha L}\right]$$

For which α is this finite?